Genaral Pi Function and the Pi number

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In this short note, we have defined a new "nested square root" function which generates usual Pi number for $x = 2$. We have given some useful identities and asymptotic formulas of the Pi-function.

I. INTRODUCTION

Studies on new methods calculating the usual **Pi-number** have been done so far, It is possible to find many works on this issue [1-10]. In this work our main interest is not getting the Pi-number, but obtaining more general Pi-function which gives us usual Pi-number for special case of the general function. This kind of definition will be helpful to get different representation of the number Pi. To be able to get this function, we will follow Viète while we are obtaining the general $\Pi(x)$ function. In 1543, Viète showed that usual Pi-number could be obtained as the following nested square root

$$\pi = \lim_{i\to\infty} (4)^{\frac{i+1}{2}} \sqrt{2 - h_i} \tag{1}$$

where

$$h_i = \sqrt{2 + h_{i-1}}, \text{ with } h_0 = 0 \tag{2}$$

One can obtain this formula using the following iterative procedure

1. First, suppose that we have a circle and its radius is $\sqrt{2}$. We can draw a square inside this circle and let the corners of that square touch the circle. Thus, the length of the one side of the square is 2. Now if we calculate the ratio of the circumference of the square to the diameter of the circle, we get

$$\pi \approx \frac{8}{2\sqrt{2}} = 2\sqrt{2} \tag{3}$$

2. Second, we can draw octagon inside the circle and let the corners of the octagon touch the circle. The radius of the circle is still $\sqrt{2}$. Thus we can easily calculate the length of the one side of the octagon as below

$$x = \sqrt{2}\sqrt{2 - \sqrt{2}} \tag{4}$$

an later we can again calculate the ratio of the circumference of the octagon to the diameter of the circle as below

$$\pi \approx 4\sqrt{2 - \sqrt{2}} \tag{5}$$

If these calculations are performed, one can easily find Eq.(1)

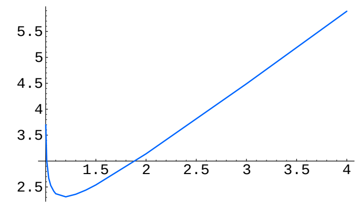

FIG. 1: Behaviour of the $\Pi(x)$

II. GENERALIZATION

In this section, we would like to write a general equation which covers Eq.(1). For this purpose, we will make the following ansatz,

$$\Pi_i(x) = (2x)^{\frac{i+1}{2}} \sqrt{x - h_i}, \; x \in (-\infty, \infty) \tag{6}$$

where

$$h_i = \sqrt{x(x-1) + h_{i-1}}, \text{ with } h_0 = 0 \tag{7}$$

This $\Pi_i(x)$ function approaches to a certain value as i goes to infinity if $x \in [1^+, \infty)$. That is to say:

$$\lim_{i\to\infty} \Pi_i(x) = \pi_x \tag{8}$$

It is easy to see that when we take $x = 2$ in Eq(6) and (7), we get Eq.(1). Eq.(6) generates interesting numbers for us. We should point out that if $z = x + iy$ $(x \neq 0)$, then we can get that $\Pi_i(z)$ approaches to same positive and negative constant complex numbers. In Table-1 we give some numerical example for $\Pi_i(x)$ and in the Figure the behavior of the *Pi-function* has been shown. To use this $\Pi-$function may be useful to produce different calculation ways for usual Pi-number. We will give some interesting properties of the *Pi-function* and later using these identities, one can to obtain different representation of the pi-number.

A. Some properties of $\Pi_i(x)$

It is easy to show that

$$\lim_{x\to\infty} \Pi_i(x) = \sqrt{2}x \left(1 + \frac{1}{8x}\right) \tag{9}$$

and we have also determined that $\Pi_i(x)$ function has a minimum value at $x \approx 1.19005$. At this value,

$$\Pi_i(1.19005) \approx 2.31383 \tag{10}$$

Another interesting properties can be written as below

$$\lim_{x\to\infty} \sqrt{\frac{x-h_i}{x-h_{i+1}}} = \sqrt{2x} \tag{11}$$

where h_j is defined as Eq.(2). In addition to above identities, we can obtain some new formulas to find number-Pi, for this purpose, one can calculate $\Pi_{i+1}(x)$ as below

$$\Pi^2_{i+1}(x) = 2x\Pi^2_i(x) + (2x)^{i+2}(h_i - h_{i+1}) \tag{12}$$

From this formula, one can easily get that when i goes to infinity, Π_{i+1} and Π_i approaches to the same π_x values given in Eq.(8). Thus, one can write the following formula while i goes to infinity

$$\pi_x = (2x)^{\frac{i+2}{2}} \sqrt{\frac{h_{i+1} - h_i}{2x-1}} \tag{13}$$

If we calculate asymptotic behavior of Eq.(13) at large x values one can obtain that

$$\pi_x = x\sqrt{\frac{4x-1}{2x-1}} \tag{14}$$

Additionally, when we look at the $h_i(x)$ function, we see that

$$h_i(x) = h_i(1-x) \tag{15}$$

from this equation, one can calculate that

$$\Pi^2_i(1-x) = (\frac{1}{x}-1)^{i+1}\Pi^2_i(x) - 2^{i+1}(1-x)^{i+1}(2x-1) \tag{16}$$

Table.1 Calculation of Pi function for different values of x. Iteration number (i) is taken as 50

x	$\Pi_i(x)$
1.001	3.7033451
1.5	2.5351046
2	**3.1415927**
2.5	3.8084662
3	4.4937674
4	5.8848462
5	7.2869301
7	10.102809
8	11.513355
20	28.469656

III. REFERENCES

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